MATH 530: Differential Geometry IV – General Relativity

22nd January 2025

Oral examination

Name: ______________________________

SCIPER No: ______________________________

Instructions:

1. This exam contains two (2) groups of questions. You will be asked to choose one of those groups and present the solution on the black board.
2. You will have 20mins to study the questions (and choose which group to solve) and then another 25mins to present the solution on the blackboard.
3. You may not use any books or other external resources while working on the questions.
4. You can use these pages to take notes/write down parts of the solution. If you need extra sheets of blank paper, please ask.

GROUP I

1. (a) Define the notion of a Lorentzian manifold $(\mathcal{M}, g)$. When is $(\mathcal{M}, g)$ called a *globally hyperbolic spacetime*?

   (b) Draw the Penrose diagram of the maximally extended Schwarzschild spacetime. Is the region $I + II$ of the diagram globally hyperbolic?

   (c) Write down the vacuum Einstein equations for $(\mathcal{M}, g)$. Show that, when $n > 1$, they are equivalent to $\mathrm{Ric} = 0$.

2. Let $\mathcal{M} = \mathbb{R} \times \mathbb{T}^3$, $g = -dt^2 + \bar{g}$, where $\bar{g}$ is a *Riemannian* metric on the torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$.

   (a) Show that the vector field ∂_t in the standard coordinate system (t, x^1, x^2, x^3) on $\mathbb{R} \times \mathbb{T}^3$ is a timelike Killing vector field.

   (b) Show that the wave operator $\Box_g$ takes the form

$$\Box_g = -\partial_t^2 + \Delta_{\bar{g}},$$

   where $\Delta_{\bar{g}}$ is the Laplace-Beltrami operator of $\bar{g}$.

   (c) Let $\psi : \mathcal{M} \to \mathbb{R}$ be a solution of the initial value problem

$$\begin{cases} \Box_g \psi = 0, \\ (\psi, \partial_t \psi)|_{t=0} = (\psi_0, \psi_1), \quad \psi_0, \psi_1 \in C^\infty(\mathbb{T}^3). \end{cases}$$

   Show that the energy

$$\mathcal{E}[\psi](\tau) = \int_{t=\tau} \Big((\partial_t \psi)^2 + |d\psi|_{\bar{g}}^2 \Big) \, \mathrm{dvol}_{\bar{g}}$$

   is *constant* in τ, where $|d\psi|_{\bar{g}}^2 \doteq \bar{g}^{ij} \partial_i \psi \partial_j \psi$.

   (d) Is it always true that $\limsup_{t \to +\infty} \sup_{x \in \mathbb{T}^3} |\psi(t, x)| < +\infty$? Prove it or find a counterexample.

GROUP II

1. Let $(\mathcal{M}^{n+1}, g)$ be a Lorentzian manifold.

   (a) Define the Levi-Civita connection ∇ of $(\mathcal{M}, g)$. Define the notion of a geodesic of $(\mathcal{M}, g)$.

   (b) Let $\gamma : (a, b) \to (\mathcal{M}, g)$ be a geodesic. Show that $g(\dot{\gamma}, \dot{\gamma})$ is constant along γ. If V is a Killing vector field of $(\mathcal{M}, g)$, show that $g(\dot{\gamma}, V)$ is constant along γ.

   (c) Let $\psi : \mathcal{M} \to \mathbb{R}$ solve $\Box_g \psi = 0$ and let V be a Killing vector field on $\mathcal{M}$. Define the current $J^{(V)}_\alpha[\psi]$ and show that it is divergence-free.

   (d) Define what is an initial data set for the Einstein vacuum equations. What is called a development of such an initial data set?

2. Let $\mathcal{M} = \mathbb{R}^{3+1}$, $g = -dt^2 + e^{2t}\big(dx^2 + dy^2 + dz^2\big)$ (this is a model of an expanding universe).

   (a) Is $(\mathcal{M}, g)$ globally hyperbolic?

   (b) Show that, for any fixed (x^1, x^2, x^3), the curve $t \to (t, x^1, x^2, x^3)$ is a timelike geodesic.

   (c) Calculate a closed formula for a *null* geodesic.

   (d) Consider the free-falling observers (i.e. timelike geodesic curves) $\gamma_1(t) = (t, 0, 0, 0)$ and $\gamma_2(t) = (t, 1, 0, 0)$. Show that there exists some $T > 0$ such that, for any $t \geq T$, no light signal (i.e. future directed null geodesic) emanating from $\gamma_1(t)$ can reach γ_2 (and vice versa). Hence, the two observers eventually become causally separated.